Stubborn Entities in Colored Toroidal Meshes

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1 Introduction

The recent developments of internet applications are showing new paradigms of interactions among users and between users and available contents. In several social applications such as Twitter, Wikipedia or Facebook the paradigm of contents creation and filtering follows a bottom up process in which contents emerge by the users "local" interactions. In this paper we focus on the problem of reaching consensus in a majority-based distributed system, i.e. achieving a global agreement by means of local interactions. The link between consensus and majority rules has been extensively treated in several works such as [4], [5], [2],[6] and [3]. In the latter the authors studied the propagation of a faulty behavior started by well placed faulty elements. The process can be described as a vertex-coloring game on graphs where the vertices are colored black (faulty) or white (non-faulty), and change their color at each round on the basis of the colors of their neighbors. We have defined and studied distributed algorithms for consensus where multicolored entities change their color on the basis of the neighboring vertices. In the first work [1] the nodes assuming directly the color of the neighbors were defined "persuadable", while in this paper we present a protocol of "stubborn" entities in which the set of colors is ordered and the entities update their color by partially incrementing their color toward the color held by their neighbors. In a socio-cognitive context of opinion sharing an entity can change one's mind under the influence of the opinion of the (simple) majority of the others (neighbors). Here we study the initial distribution of colors producing a monochromatic configuration within a finite time.

2 Notation and Definitions

We study the interaction of *stubborn* entities in a specific topology of the system:

Definition 1. *A toroidal mesh* $T : (V, E)$ *of* $m \times n$ *vertices is a mesh where each vertex* $v_{i,j}$ *(* $0 \leq i \leq m-1$ *and* $0 \leq j \leq n-1$*) is connected to the four vertices* $v_{(i-1) \bmod m, j}$ *,* $v_{(i+1) \bmod m, j}$ *,* $v_{i,(j-1) \bmod n}$ *and* $v_{i,(j+1) \bmod n}$.

Let $\mathcal{C} = \{1, \ldots, k\}$ be a finite set of colors. A coloring of a torus T is a function $r : V \rightarrow \mathcal{C}$. A **bi-colored torus** is a coloring of T that uses two colors and a **multi-colored torus** is a coloring of T that uses more than two colors. Let $N(x)$ denote the neighborhood of any vertex x in V; we have that $|N(x)| = 4$. Given a coloring r of V, we can define the following irreversible simple rule **(StubSM-Protocol)**:

for all $x \in T$ **do**
  let $a, b, c, d \in N(x)$
  if $(r(a) = r(b) > r(x)) \wedge ((r(c) \neq r(d)) \vee (r(c) = r(d) > r(x)))$ **then**
    $r(x) \leftarrow r(x) + 1$

We denote the subset of T of all k-colored vertices by S^k, and the set of its k-colored vertices by V^k ($k \in \mathcal{C}$). The recoloring process represents the dynamics of the system. Depending on the initial coloring of T, we get different dynamics. Among the possible initial configurations (i.e. assignments of colors) we are interested in those leading to a monochromatic coloring, so called dynamos. Formally,

Definition 2. *The set S^k is a **dynamo** if an all k-color configuration is reached from S^k in a finite number of steps under the **StubSM-Protocol**.*

Finally we need to introduce the following definitions.

Definition 3. *A **k-block** B^k is a connected subset of T made up of vertices of the same color k each of which has at least* **two neighbors in B^k.**

Note that vertices in B^k will never change their color. For example, B^k can be a k-colored column (row), any 2×2 submatrix of consecutive rows and columns that we call a **window**, or any k-colored cycle such that $v_{i,j}, v_{i,j+1}, \ldots, v_{i,j'}$, $v_{i-1,j'}, \ldots, v_{i',j'}$, $v_{i',j'-1}, \ldots v_{i',j}$, $v_{i'+1,j}, \ldots v_{i,j}$ that we call a **frame**.

Definition 4. *A **non-k-block** NB^k is a connected subset of T made up of vertices of colors in $\mathcal{C} \setminus \{k\}$ each of which has at least* **three neighbors in NB^k.**

This definition implies that every vertex in NB^k has at most one k-colored neighbor, namely, vertices in NB^k will never recolor by k color. For example, two consecutive rows or columns of vertices not colored by k constitute a non-k-block in a toroidal mesh.

3 Bounds to the size of a dynamo

In this paper, we are interested in determining the minimum size dynamo under the **StubSM-Protocol** for a multi-colored toroidal mesh. This is obtained by first computing a lower bound to the size and then an upper bound close to the lower bound. These bounds can be derived by a reduction to the *bi*-colored case. We define a polynomial time transformation $\phi : \mathcal{C} \rightarrow \mathcal{C}$ such that $\phi(i) = 1$, for $i = 1, \ldots, k-1$, and $\phi(k) = 2$. This transformation allows us to map a multi-colored torus into a bi-colored torus (where 1 and 2 correspond to colors white and black, respectively). Moreover under transformation ϕ, a *non-k*-block corresponds to a *simple white block* of [3].

Proposition 1. *A lower bound to the size of a dynamo in a bi-colored torus under the (reversible) simple majority rule is a lower bound to the size of a dynamo in a multi-colored torus under the **StubSM-Protocol**.*

Indeed a lower bound consists in the smallest size of S^2 (initial set of black vertices) such that no simple white blocks can arise in the first problem, and in the smallest size of S^k such that no *non-k*-blocks can arise in the second problem. Because of the correspondence between a *non-k*-block and a simple white block the claim follows. Therefore we derive (see Theorem 9 of [3]):

Theorem 1. *Let S^k be a dynamo for a colored toroidal mesh of size $m \times n$. We have*

- *(i)* $m_{S^k} \geq m-1,\ n_{S^k} \geq n-1$
- *(ii)* $|S^k| \geq m+n-2$.

Proposition 2. *An upper bound to the size of a dynamo in a bi-colored torus under the (irreversible) strong majority rule is an upper bound to the size of a dynamo in a multi-colored torus under the* ***StubSM-Protocol****.*

Indeed in order to establish an upper bound to the size of S^2, no *strong white blocks* of [3] have to arise and the successive derived black sets of vertices have to contain the set V of all the vertices at the end of the process, in the first problem. Similarly, to obtain an upper bound to the size of S^k, no i-blocks have to arise and successive derived k-colored sets of vertices have to contain the set V of all the vertices at the end of the process, in the second problem. We have that: a) strong white blocks correspond to i-blocks; b) irreversible strong majority rule is more restrictive than **StubSM-Protocol**: indeed, under irreversible strong majority rule, a vertex recolors itself if there are three vertices in its neighborhood having the same color, whereas under the **StubSM-Protocol** two neighbors with the same color are requested (and the others with different colors). Because of a) and b) the claim follows. Therefore we get (see Theorem 8 of [3]):

Theorem 2. *Let S^k be a dynamo for a colored toroidal mesh of size $m \times n$. Then $|S^k| \geq \lceil m/3 \rceil (n+1)$.*

4 A minimum size dynamo

Theorem 2 establishes an upper bound far from the lower bound determined in Theorem 1. In this section we derive a dynamo of minimum size.

Lemma 1. *Let S^k be a dynamo. Then, $T - S^k$ does not contain any h-block, with $h \in \mathcal{C} \setminus \{k\}$.*

Let S^k be made up of the first row and column in the torus. Then $|S^k| = m+n-1$ that is close to the lower bound in Theorem 1. In this case, no h-colored column or h-colored row can arise, but this does not hold for h-colored window or a h-colored frame, with $h \in \mathcal{C} \setminus \{k\}$. As a consequence we require that for every 2×2 window in T, $r(v_{i,j}) \neq r(v_{i+1,j+1})$ and $r(v_{i,j+1}) \neq r(v_{i+1,j})$; otherwise $r(v_{i,j}) = r(v_{i+1,j+1}) = k$ ($r(v_{i,j+1}) = r(v_{i+1,j}) = k$). This requirement does not forbid that during the recoloring process a $h - block$ can appear as Figure 1 illustrates.

The following theorem considers the consequences of the recoloring dynamic that can produce a certain h-block.

$$\begin{array}{ccccccccccccc}
8&8&8&8&8&8&\dashrightarrow&8&8&8&8&8&8\\
8&7&7&3&4&6&\dashrightarrow&8&8&8&6&7&8\\
8&1&4&3&7&5&\dashrightarrow&8&5&\mathbf{4}&\mathbf{4}&7&8\\
8&1&4&3&2&4&\dashrightarrow&8&1&\mathbf{4}&\mathbf{4}&2&7\\
8&1&4&3&2&4&\dashrightarrow&8&6&4&3&3&8
\end{array}$$

Fig. 1. An Example in which a block emerges after five steps.

$$\begin{array}{ccccc}
6&6&6&6&6\\
6&5&5&5&5\\
6&4&4&4&4\\
6&1&1&1&1\\
6&3&3&3&3\\
6&5&5&5&5
\end{array}
\qquad\qquad
\begin{array}{ccccc}
7&7&7&7&7\\
7&6&6&6&6\\
7&5&5&5&5\\
7&1&1&1&1\\
7&6&6&6&6
\end{array}$$

Fig. 2. Two examples of dynamos when m is even and odd, respectively.

Theorem 3. *Let a_W, b_W, c_W, d_W be the vertices of any 2×2 window with $r(a_W) \leq r(b_W) \leq r(c_W) \leq r(d_W)$ and i and j be the number of recoloring of a_W and d_W, respectively under the* ***StubSM-Protocol****. No h-block can appear if $i - j < r(d_W) - r(a_W)$, with $h \in \mathcal{C} \setminus \{k\}$.*

By this theorem and considerations about the recoloring pattern due to the choice of S^k, we can derive an initial distribution of the remaining colors leading to the k monochromatic configuration. Due to space limit in Figure 2 we only give two examples of dynamos when m is even and odd, respectively.

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